

GENERALIZED DERIVATIONS WITH CENTRAL VALUES ON LIE IDEALS

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ABSTRACT. let R be a prime ring of $\text{char} R \neq 2$, H a generalized derivation and L a noncentral lie ideal of R . We show that if $l^s H(l) l^t \in Z(R)$ for all $l \in L$, where $s, t \geq 0$ are fixed integers, then $H(x) = bx$ for some $b \in C$, the extended centroid of R , or R satisfies S_4 . Moreover, let R be a 2-torsion free semiprime ring, let $A = O(R)$ be an orthogonal completion of R and $B = B(C)$ the Boolean ring of C . Suppose $([x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t) \in Z(R)$ for all $x_1, x_2 \in R$, where $s, t \geq 0$ are fixed integers. Then there exists idempotent $e \in B$ such that $H(x) = bx$ on eA and the ring $(1 - e)A$ satisfies S_4 .

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1. INTRODUCTION

Let R be an associative ring with center $Z(R)$. Recall that an additive map $d : R \rightarrow R$ is called derivation if $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. Many results in literature indicate that global structure of a prime(semiprime) ring R is often lightly connected to the behaviour of additive mappings defined on R . A well-known result of Herstein [6] stated that if d is a nonzero derivation of a prime ring R such that $d(x)^n \in Z(R)$ for all $x \in R$, then R satisfies S_4 , the standard identity in four variables. Herstein's result was extended to the case of Lie ideals of prime rings by Bergen and Carini [2]. Some articles was studied derivation with central values on Lie ideals [4, 10]. Recently, Dhara [5] studied the more generalized situation when $l^s d(l) l^t \in Z(R)$, for all $l \in L$, the noncentral Lie ideal of R , where $s, t \geq 0$ are some fixed integers.

Here we will consider the same situation in case the derivation d is replaced by generalized derivation H . More specifically an additive map $H : R \rightarrow R$ is called generalized derivation if there is a derivation d of R such that $H(xy) = H(x)y + xd(y)$, for all $x, y \in R$.

Throughout the paper we use the standard notation from [1]. In particular, we denote by Q the two sided Martindale quotient of prime(semiprime) ring R and C the center of Q . We call C the extended centroid of R .

The main results of this paper are as follows:

Theorem 1.1. *Let R be a prime ring of $\text{char} R \neq 2$, H generalized derivation and L a noncentral Lie ideal of R . Suppose $l^s H(l) l^t \in Z(R)$ for all $l \in L$, where $s, t \geq 0$, are fixed integers. Then $H(x) = bx$ for some $b \in C$, the extended centroid of R , or R satisfies S_4 .*

Key words and phrases. generalized derivation, prime ring, Martindale quotient ring.

When R is a semiprime ring, we prove:

Theorem 1.2. *let R be a 2-torsion free semiprime ring with generalized derivation H . Consider $[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t \in Z(R)$ for all $x_1, x_2 \in R$, where $s, t \geq 0$ are fixed integers. Further, let $A = O(R)$ be the orthogonal completion of R and $B = B(C)$ where C the extended centroid of R . Then there exists idempotent $e \in B$ such that $H(x) = bx$ on eA and the ring $(1 - e)A$ satisfies S_4 .*

2. PROOF OF THE MAIN RESULTS

The following results are useful tools needed in the proof of the main results.

Lemma 2.1. *Every generalized derivation H on a dense right ideal of prime(semiprime) ring R can be uniquely extended to a generalized derivations of Q . Also can be write in the form $H(x) = bx + d(x)$ for some $b \in Q$, all $x \in Q$ and a derivation d of Q [11].*

Lemma 2.2. (see [8, Lemma 2] and [3, Lemma 1]). *Let R be a prime ring of char $R \neq 2$, L be a noncentral Lie ideal of R and I be the ideal of R generated by $[L, L]$. Then $I \subseteq L + L^2$ and $[I, I] \subseteq L$.*

Theorem 2.3. (Kharchenko [7]). *Let R be a prime ring, d a nonzero derivation of R and I a nonzero ideal of R . If I satisfies the differential identity*

$$f(r_1, r_2, \dots, r_n, d(r_1), d(r_2), \dots, d(r_n)) = 0,$$

for any $r_1, r_2, \dots, r_n \in I$, then one of the following holds:

(i) *I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \dots, r_n, x_1, x_2, \dots, x_n) = 0.$$

(ii) *d is Q -inner, that is, for some $q \in Q$, $d(x) = [q, x]$ and I satisfies the generalized polynomial identity*

$$f(r_1, r_2, \dots, r_n, [q, r_1], [q, r_2], \dots, [q, r_n]) = 0.$$

We establish the following technical results required in the proof of Theorem 1.1.

Lemma 2.4. *Let $R = M_k(F)$ be a ring of all $k \times k$ matrices over a field F where $k \geq 3$. Suppose $b[x_1, x_2] + [x_1, x_2]c \in Z(R)$ for some $b, c \in R$ and all $x_1, x_2 \in R$. Then $b, c \in F \cdot I_k$.*

Proof. Let $b = (b_{ij})_{k \times k}$, $c = (c_{ij})_{k \times k}$. Putting $x_1 = e_{11}$ and $x_2 = e_{12}$, we obtain $b[x_1, x_2] + [x_1, x_2]c = be_{12} + e_{12}c$. Since rank of $b[x_1, x_2] + [x_1, x_2]c \leq 2$, it can not be invertible. This implies $be_{12} + e_{12}c = 0$. Left and right multiplying by e_{12} , we get

$$\begin{aligned} 0 &= e_{12}(be_{12} + e_{12}c) = b_{21}e_{12}, \\ 0 &= (be_{12} + e_{12}c)e_{12} = c_{21}e_{12}. \end{aligned}$$

This implies that $c_{21} = b_{21} = 0$. Thus for any $i \neq j$, $b_{ij} = c_{ij} = 0$. That is, b and c are diagonal. Let $b = \sum_{i=1}^k b_{ii}e_{ii}$. For any F -automorphism θ of R b^θ enjoys the same property as b does, namely, $b^\theta[x_1, x_2] + [x_1, x_2]c^\theta$ is zero or invertible, for every $x_1, x_2 \in R$. Hence b^θ must be diagonal. Then for each $j \neq 1$,

$$(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^k b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j},$$

is diagonal. Therefore, $b_{jj} = b_{11}$ and so $b \in F \cdot I_k$. Similarly, we conclude $c \in F \cdot I_k$. \square

Lemma 2.5. *Let $R = M_k(F)$ be a ring of all $k \times k$ matrices over a field F of $\text{char} F \neq 2$, where $k \geq 3$. Suppose $[x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t \in Z(R)$, for some $b, c \in R$ and all $x_1, x_2 \in R$ where $s, t \geq 0$ are fixed integers such that $s+t \neq 0$. Then $b, c \in F \cdot I_k$.*

Proof. Let $b = (b_{ij})_{k \times k}$, $c = (c_{ij})_{k \times k}$ and set

$$f(x_1, x_2) = [x_1, x_2]^s(b[x_1, x_2] + [x_1, x_2]c)[x_1, x_2]^t.$$

Putting $x_1 = e_{11}$, $x_2 = e_{12} - e_{21}$, we obtain $[x_1, x_2] = e_{12} + e_{21}$ and $[x_1, x_2]^n = e_{11} + e_{22}$ for $n \geq 2$. So we have four cases:

Case 1. $s = t = 1$. We get

$$f(x_1, x_2) = (b_{21} + c_{12})e_{11} + (b_{12} + c_{21})e_{22} + (b_{22} + c_{11})e_{12} + (b_{11} + c_{22})e_{21}.$$

Case 2. $s = 0$ and $t = 1$. We get

$$f(x_1, x_2) = (b_{11} + c_{22})e_{11} + (b_{22} + c_{11})e_{22} + (b_{12} + c_{21})e_{12} + (b_{21} + c_{12})e_{21} + \sum_{i=3}^k b_{i1}e_{i1} + \sum_{i=3}^k b_{i2}e_{i2}.$$

Case 3. $s = 1$ and $t = 0$. We get

$$f(x_1, x_2) = (b_{22} + c_{11})e_{11} + (b_{11} + c_{22})e_{22} + (b_{21} + c_{12})e_{12} + (b_{12} + c_{21})e_{21} + \sum_{i=3}^k c_{1i}e_{i1} + \sum_{i=3}^k c_{2i}e_{i2}.$$

Case 4. $s, t \geq 2$. We obtain

$$f(x_1, x_2) = (b_{12} + c_{21})e_{11} + (b_{21} + c_{12})e_{22} + (b_{11} + c_{22})e_{12} + (b_{22} + c_{11})e_{21}.$$

In each cases, since $\text{rank of } f(x_1, x_2) \leq 2$, $f(x_1, x_2) = 0$. Thus

$$b_{12} = -c_{21} \quad \text{and} \quad b_{21} = -c_{12},$$

and so for any $i \neq j$ we have

$$(1) \quad b_{ij} = -c_{ji}.$$

Now putting $x_1 = e_{11}$, $x_2 = e_{12} + e_{21}$, we have $[x_1, x_2]^n = (-1)^{n/2}(e_{11} + e_{22})$ if n is even and $(-1)^{(n-1)/2}(e_{12} - e_{21})$ if n is odd. Four cases may be occurred:

Case 1. s and t are even. We get

$$f(x_1, x_2) = \pm((-b_{12} + c_{21})e_{11} + (b_{21} - c_{12})e_{22} + (b_{11} + c_{22})e_{12} + (-b_{22} - c_{11})e_{21}).$$

Case 2. s and t are odd. We get

$$f(x_1, x_2) = \pm((-b_{21} + c_{12})e_{11} + (b_{12} - c_{21})e_{22} + (-b_{22} - c_{11})e_{12} + (b_{11} + c_{22})e_{21}).$$

Case 3. s is even and t is odd. We get

$$f(x_1, x_2) = \pm((-b_{11} - c_{22})e_{11} + (-b_{22} - c_{11})e_{22} + (-b_{12} + c_{21})e_{12} + (-b_{21} + c_{12})e_{21}).$$

Case 4. s is odd and t is even. We get

$$f(x_1, x_2) = \pm((-b_{22} - c_{11})e_{11} + (-b_{11} - c_{22})e_{22} + (b_{21} - c_{12})e_{12} + (b_{12} - c_{21})e_{21}).$$

In each cases, since $\text{rank of } f(x_1, x_2) \leq 2$, $f(x_1, x_2) = 0$. Thus

$$b_{12} = c_{21} \quad \text{and} \quad b_{21} = c_{12},$$

and so for any $i \neq j$ we have

$$(2) \quad b_{ij} = c_{ji}.$$

(1) and (2) imply that b and c are diagonal. So we apply the same argument used in the proof of Lemma 2.4. Hence $b, c \in F \cdot I_k$. \square

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Since $\text{char} R \neq 2$ and L is noncentral Lie ideal, by Lemma 2.2 there exists an ideal I of R such that $0 \neq [I, I] \subseteq L$ and $[L, L] \neq 0$. Hence, without loss of generality, we may assume $L = [I, I]$. Thus I satisfies the generalized differential identity

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t \in Z(R).$$

Let Q be the two sided Martindale quotient ring and C the extended centroid of R . By [11] I and Q satisfy the same differential identities, thus we may assume

$$[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t \in Z(R),$$

for all $x_1, x_2 \in Q$. By Lemma 2.1 we may assume $H(x) = bx + d(x)$ for some $b \in Q$, all $x \in Q$ and d a derivation of Q . Hence Q satisfies

$$[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t \in Z(R).$$

This is a polynomial identity. Hence there exists a field F such that $Q \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F , where $k > 1$. Moreover Q and $M_k(F)$ satisfy the same polynomial identity [9]. Hence we have

$$(3) \quad [x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t \in Z(M_k(F)).$$

Now consider two cases.

case 1. d is a Q -inner derivation. In this case, there exists an element $p \in Q$ such that $d(x) = [p, x]$ for all $x \in M_k(F)$, then (3) becomes

$$[x_1, x_2]^s (b[x_1, x_2] + [p, [x_1, x_2]])[x_1, x_2]^t \in Z(M_k(F)).$$

So

$$[x_1, x_2]^s ((b+p)[x_1, x_2] - [x_1, x_2]p)[x_1, x_2]^t \in Z(M_k(F)),$$

for all $x_1, x_2 \in M_k(F)$. In this case if $k \geq 3$ and $s = t = 0$, then by Lemma 2.4 we have $-p, b+p \in F \cdot I_k$. Also for $k \geq 3$ and $s+t \neq 0$, Lemma 2.5 implies $-p, b+p \in F \cdot I_k$. Then $b \in F \cdot I_k$, and so $d(x) = 0$. Hence $H(x) = bx$ for all $x \in M_k(F)$. So by [9] for all $x \in R$ we have $H(x) = bx$. If $k = 2$, then R satisfies S_4 .

case 2. d is not a Q -inner derivation. In this case we have

$$[[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t, x_3] = 0,$$

for all $x_1, x_2, x_3 \in M_k(F)$.

Then by Theorem 2.3 we have

$$[[x_1, x_2]^s (b[x_1, x_2] + [x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = 0,$$

for all $x_1, x_2, x_3, x_4, x_5 \in M_k(F)$. In particular, $M_k(F)$ satisfies its blended component

$$[[x_1, x_2]^s ([x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = 0.$$

If $k \geq 3$, then by choosing

$$x_1 = e_{ij}, \quad x_2 = e_{ji}, \quad x_3 = e_{ik}, \quad x_4 = e_{ij}, \quad x_5 = 0,$$

for all $i \neq j \neq k$, we get

$$0 = [[x_1, x_2]^s([x_4, x_2] + [x_1, x_5])[x_1, x_2]^t, x_3] = e_{ik},$$

which is a contradiction. Thus $k = 2$, that is, R satisfies S_4 . \square

Now let R be a semiprime orthogonally complete ring with extended centeroid C . The notations $B = B(C)$ and $\text{spec}(B)$ denotes Boolean ring of C and the set of all maximal ideal of B , respectively. It is well known that if $M \in \text{spec}(B)$ then $R_M = R/RM$ is prime [1, Theorem 3.2.7]. We use the notations Ω - Δ -ring, Horn formulas and Hereditary formulas. We refer the reader to [1, pages 37, 38, 43, 120] for the definitions and the related properties of these objects.

We establish the following technical result required in the proof of Theorem 1.2.

Lemma 2.6. [1, Theorem 3.2.18]. *Let R be an orthogonally complete Ω - Δ -ring with extended centroid C , $\Psi_i(x_1, x_2, \dots, x_n)$ Horn formulas of signature Ω - Δ , $i = 1, 2, \dots$ and $\Phi(y_1, y_2, \dots, y_m)$ a Hereditary first order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \dots, a_n) \in R^{(n)}$, $\vec{c} = (c_1, c_2, \dots, c_m) \in R^{(m)}$. Suppose $R \models \Phi(\vec{c})$ and for every $M \in \text{spec}(B)$ there exists a natural number $i = i(M) > 0$ such that*

$$R_M \models \Phi(\phi_M(\vec{c})) \implies \Psi_i(\phi_M(\vec{a})),$$

where $\phi_M : R \rightarrow R_M = R/RM$ is the canonical projection. Then there exists a natural number $k > 0$ and pairwise orthogonal idempotents $e_1, e_2, \dots, e_k \in B$ such that $e_1 + e_2 + \dots + e_k = 1$ and $e_i R \models \Psi_i(e_i \vec{a})$ for all $e_i \neq 0$.

we denote $O(R)$ the orthogonal completion of R which is defined as the intersection of all orthogonally complete subset of Q containing R .

Now we can prove Theorem 1.2.

Proof of Theorem 1.2. By assumption we have R satisfies

$$[[x_1, x_2]^s H([x_1, x_2])[x_1, x_2]^t, x_3] = 0.$$

By Lemma 2.1 the generalized derivation H can be extended uniquely to the generalized derivation on Q , moreover, we may assume $H([x_1, x_2]) = b[x_1, x_2] + d([x_1, x_2])$, for some $b \in Q$, all $x_1, x_2 \in Q$ and d a derivation of Q . Hence Q satisfies

$$[[x_1, x_2]^s (b([x_1, x_2] + d([x_1, x_2]))[x_1, x_2]^t, x_3] = 0.$$

According to [1, Theorem 3.1.16] $d(A) \subseteq A$ and $d(e) = 0$ for all $e \in B$. Therefore, A is an orthogonally complete Ω - Δ -ring, where $\Omega = \{o, +, -, \cdot, d\}$. Consider formulas

$$\Phi = (\forall x_1)(\forall x_2) \| [[x_1, x_2]^s (b[x_1, x_2] + d([x_1, x_2])[x_1, x_2]^t, x_3] = 0 \|,$$

$$\Psi_1 = (\forall x) \| H(x) = bx \|,$$

$$\Psi_2 = (\forall x_1)(\forall x_2)(\forall x_3)(\forall x_4) \| S_4(x_1, x_2, x_3, x_4) = 0 \|.$$

We can easily check that Φ is a hereditary first order formula and $\neg\Phi, \Psi_1, \Psi_2$

are Horn formulas. So using Theorem 1.1, all conditions of Lemma 2.6 are fulfilled. Hence there exist two orthogonal idempotents e_1 and e_2 such that $e_1 + e_2 = 1$. If $e_i \neq 0$, then $e_i A \models \Psi_i$, $i = 1, 2$. This complete the proof. \square

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